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Accounting for mathematicians' priorities in mathematics courses for secondary teachers

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ABSTRACT

Recent studies suggest that change is needed in undergraduate mathematics capstone courses for prospective secondary teachers. One promising but infrequently used strategy for improvement is to incorporate tasks that explicitly focus on pedagogical content knowledge (PCK). Expectancy-value theory provides an account for why instruction of these courses does not more regularly employ this strategy. To make this argument, this paper uses an interview-based study of 9 mathematicians that investigated the process of prioritizing tasks and goals for these courses. As the study found, these mathematicians valued developing teachers' PCK. However, they were unconfident of their ability to teach with tasks and goals focused on developing teachers' PCK relative to more purely mathematical tasks and goals. The central implication is that interventions in mathematicians' teaching must take into account the possibility that it may be just as important to improve confidence and resources as it is to change values.

1. Accounting for priorities of mathematics courses for secondary teachers using expectancy-value theory

Multiple studies on secondary mathematics teacher education suggest that teachers do not perceive their mathematics courses as applicable to teaching practice (Goulding et al., 2003; Ticknor 2012; Wasserman et al., 2018; Zazkis and Leikin 2010). Thus, even if applications could be made, it seems unlikely that teachers would leverage them if they believe there is no connection (Lobato 2012; Wasserman et al. 2018). The above literature suggests that change is needed in mathematics courses for prospective teachers.

One approach for shifting these courses is to understand and improve these courses' instruction, which is largely enacted by mathematics faculty. However, there are few studies of the teaching practice of mathematics faculty (Speer et al., 2010).

Effective efforts to shift a person's teaching practice begin with understanding that person's practice, including their choices (e.g., Schoenfeld 2010). Expectancy-value theory, holds that persons are most likely to choose activities that they both *value* and *expect to do well* in – and that they may not choose a valued activity if they have little confidence that they will do well at it. For example, even if teachers value an activity because it aligns with their goals, they may not follow through with that activity if they expect to use that activity poorly.

In this paper, I report on an interview study in which 9 mathematicians in the US were asked about their expectancy and value regarding competing goals for undergraduate mathematics capstone courses that enroll future secondary teachers. By "capstone course", I refer to courses designed specifically for teachers, as opposed to those in which prospective teachers learn mathematics along with mathematics concentrators. Prospective secondary mathematics teachers often take these courses at the end of their undergraduate study in the US; they are intended to promote connections between university and secondary mathematics, and to strengthen understanding of secondary mathematics (Conference Board of the Mathematical Sciences 2012). By "mathematician", I

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refer to a person with a mathematics doctoral degree, who is employed as a faculty member in a department of mathematics, and whose professional engagements include teaching and scholarship in mathematics.

This study was guided by the questions: What approaches to goals and tasks do mathematicians value for mathematics capstone courses for prospective secondary teachers? What expectancy do mathematicians hold for these goals and tasks? What factors influence value and expectancy? I used interviews to provide insight into ways that mathematicians may determine value of an instructional task or goal, factors that influence their expectancy in carrying out an instructional task or goal, and how these relate.

2. Background

I use results of prior literatureto make the case that, in designing capstone courses for future secondary teachers, there is much promise in incorporating an approach that develops teachers' PCK– and that implementation may come with challenges. This argument sets the stage for this study, which features instructional goals and tasks addressing PCK.

2.1. Empirical results on the effectiveness of secondary teachers' mathematical preparation

Pure mathematics is a staple of secondary mathematics teachers' education (e.g., Ferrini-Mundy and Findell 2004). However, many teachers find their mathematical preparation disconnected from teaching, as evidenced by Zazkis and Leikin's (2010) survey of 52 practicing teachers, and Goulding, Hatch and Rodd's (2003) survey of 173 prospective teachers from the UK. Ticknor (2012) interviewed five prospective secondary teachers and a mathematician instructor, and found that while the instructor saw connections between abstract algebra and teaching school mathematics and the teachers were engaged in the course, the teachers still perceived the course as irrelevant to teaching. Wasserman et al. (2018) interviewed 14 secondary teachers and found similar results for real analysis. Thus, though there are theoretical arguments for providing experiences in pure mathematics, it has been hard to establish these benefits in practice. Wasserman et al. (2018) and Ticknor (2012), concurring with other scholars (e.g., Stylianides and Stylianides 2014), concluded that these benefits may be more likely to occur if mathematical content was more explicitly and intentionally situated in teaching practice – in other words, for course instruction to focus on the development of *pedagogical content knowledge (PCK)*, which I discuss next.

2.2. Pedagogical content knowledge (PCK)

2.2.1. Conceptualizing PCK

The literature in teacher education conceptualizes "pedagogical content knowledge" in a variety of ways (Depaepe et al., 2013). However, there is common ground: across studies, PCK "deals with teachers' knowledge, connects content and pedagogy, is specific to teaching particular subject matter, and content knowledge is an important and necessary prerequisite" (p. 22). I take this common ground as a conception of PCK for this study. To further specify PCK, I turn to Shulman's work, which is almost universally cited to introduce the concept of PCK (Depaepe et al., 2013). Shulman (1987) pointed to "transformation" to "distinguish a teacher from non-teacher peers" (p. 15), including content experts; he defined transformation as interpreting instructional materials, representing ideas, adapting representations to learners, and choosing pedagogical methods or models. Furthermore, he specified that representation involved selecting "analogies, metaphors, examples, demonstrations, explanations" (p. 15).

2.2.2. Tasks and goals intended for developing PCK

A task or goal used in this study represented PCK if it situated mathematical knowledge in an act of transformation as described above, with a particular emphasis on representation. The scope of representation was chosen because it was at once broad enough to encompass consequential instructional decision making in secondary mathematics (e.g., Lloyd and Wilson 1998), consistent with conceptualizations of PCK at secondary level (e.g., Krauss et al., 2008; Tatto, Schwille, Senk, Ingvarson, Peck, and Rowley 2008), and feasible in the context of a capstone course. Examples of tasks intended for developing PCK include the Green Task, Blue Task, and Orange Task shown in Fig. 3. The Green Task asks for selecting an example; reasoning on this task applies knowledge of factoring to infer that discussion of different solution paths is best supported by selecting the radicand with the most possible perfect square factors, which is in this case, 72. The Orange and Blue Tasks involve constructing explanations for learners.

2.2.3. Links to teaching and learning at the secondary level

It is important to develop teachers' PCK. At the secondary level, PCK is linked to mathematics teaching quality and student outcomes, and more so than content knowledge alone. Baumert et al.'s (2010) study, based on a stratified random sample of schools in Germany involving over 4000 secondary level students, found that German teachers' performance on a PCK measure predicted teaching quality and student achievement more than teachers' performance on a content knowledge measure did. Based on a review of research on mathematical knowledge for teaching at the secondary level, McCrory et al. (2012) concluded that secondary mathematics teachers need knowledge beyond what they will teach, including PCK. These results give credence to the value of developing teachers' PCK.

Yet, there is evidence that capstone courses for prospective secondary teachers do not focus on developing PCK. In Cox et al.'s (2013) survey of instructors of capstone courses for prospective secondary teachers, only 18% of participants cited "pedagogical concerns" as a topic covered, in comparison to 66.5% addressing "advanced mathematical topics". Corroborating this finding is that current texts at best weakly support teaching for the development of PCK in secondary teacher education; relatively few tasks situate mathematics in teaching scenarios, and the vast majority of tasks focus on pure mathematics (Lai and Patterson 2017).

2.3. Approaches to mathematical preparation for teaching at the secondary level

There is considerable debate as to the mathematics to develop in secondary teacher education. As Baumert et al. (2010) reviewed:

> There is disagreement on the necessarybreadth and depth of teachers' mathematical training (cf. Ball & Bass, 2003; Deng, 2007; Shulman, 1987): Do secondary mathematics teachers need a command of the academic research knowledge taught in the mathematics departments of universities? Or is it mathematical knowledge for teaching that matters, integrating both mathematical and instructional knowledge, as taught at schools of education? (p. 136)

Baumert et al.'s observation captures the distinction between PCK and purer forms of mathematics content knowledge, which is echoed in various theoretical frameworks for knowledge for teaching at the secondary level. These include frameworks of the projects Cognitive Activation in the Classroom (COACTIV; Krauss et al. 2008), Teacher Education Study in Mathematics (TEDS-M; Tatto et al. 2008), Knowledge of Algebra for Teaching (KAT; McCrory et al. 2012); and Mathematical Knowledge for Teaching Geometry (MKT-G; Herbst and Kosko 2014). It may be that an ideal program would draw from multiple approaches; knowledge similar to that from an advanced standpoint may support the development of PCK (Krauss et al. 2008). I now summarize the arguments for three dominant approaches to mathematical preparation at the secondary level (cf. Wasserman et al. 2018), how they might be substantiated by developers of research frameworks for mathematical knowledge for teaching at the secondary level, and following this, how these approaches relate to mathematicians' values and challenges regarding mathematics teacher education.

At the secondary level, mathematical preparation for teaching may include (1) "mathematics from an advanced standpoint", which has been interpreted to include underlying structure and conceptual understanding of foundational mathematical ideas (cf. Baumert et al. 2010; Klein (1932)/1908; Usiskin et al. 2003). It is beyond the scope of this article to detail the various conceptualizations. Of most relevance here is that policy writers have advocated for this knowledge to support conceptual understanding and connecting mathematics across the curriculum (e.g., CBMS 2012), and that this interpretation has been used to conceptualize mathematical knowledge for teaching at the secondary level (e.g., COACTIV, see Krauss et al. 2008). The Mathematical Understandings for Secondary Teaching (MUST) authors also argued for the importance of conceptual understanding and curricular connections, as students' knowledge of these is arguably strongly influenced by their teachers', and teachers must provide students a foundation for mathematics encountered later (Heid et al., 2015).

Content knowledge in teacher preparation may also include (2) connections to disciplinary or tertiary content and practices (Murray and Star 2013). Such an approach can give intellectual purpose to school mathematics by demonstrating how school mathematics leads to compelling problems of the discipline (CBMS 2012). For instance, the KAT and TEDS-M frameworks incorporated abstract algebra and linear algebra, so that ``a teacher [has] some perspective on the trajectory and growth of mathematical ideas beyond school algebra" (McCrory et al. 2012, p. 597).

And, content knowledge may include (3) mastery of conventional mathematics at the secondary level. A notion spanning back more than a century (Hill et al., 2007), which informs framework and assessment design (e.g., TEDS-M) and is codified by policies (e.g., No Child Left Behind 2002), is that teachers should know the mathematics of the level of instruction and the equivalent of at least one degree level higher. I use "conventional" to refer to a "conventional" course of study (McCrory et al. 2012, p. 598). The Violet Task (Fig. 3) represents this approach.

These approaches to mathematics teacher education, along with an approach that (4) develops teachers' PCK, are summarized in Table 1.

These approaches can overlap, especially (1) and (2), but they also contain distinct ideas. To operationalize the differences between (1) and (2) for this study, I drew on Murray and Star's (2013) review of capstone courses for secondary teachers, which distinguishes specifically between approaches (1) and (2). Murray and Star described approach (1) as one where "secondary mathematics topics are covered at a level of depth and rigor suitable for undergraduate mathematics students" (p. 1298). The Pink Task (Fig. 3) represents this approach; it involves only properties used in secondary mathematics.

In contrast, approach (2) aims for teachers to "see how concepts and methods of proof in real analysis [or other tertiary content] are directly applicable to the mathematics they will be teaching" (Murray and Star 2013, p. 1298). This approach positions secondary mathematics ideas as consequences of those of tertiary mathematics. The Yellow Task (Fig. 3) represents this approach: it positions a phenomenon that may be encountered in secondary mathematics as a consequence of the tertiary concept of multi-valued exponential functions. The Yellow Task does not require pedagogical reasoning, such as explanation for secondary students. The sequences of equations are straightforward, and the explanation asked for (using multi-valued exponential functions) is suitable only at the tertiary level.

2.4. Values and challenges in mathematicians' views of the mathematics that prospective secondary teachers should learn

Mathematicians who teach prospective or practicing mathematics teachers are "de facto" mathematics educators. (Leikin et al., 2017). In this capacity, mathematicians have repeatedly expressed the need for all teachers to have strong mathematical content (such as axioms, theorems, or proof techniques) and meta-mathematical understanding (such as appreciating the importance of precision and rigor, or recognizing the role of axioms and definitions) (e.g., Bass 2005; Cuoco 2001; Wu 2011). In these broad strokes, mathematicians and mathematics education researchers appear to agree (e.g., Heid et al., 2015; McCrory et al. 2012; Tatto et al. 2008). In the details are disagreements, even among mathematicians.

Table 1
Approaches to mathematics teacher education, with illustrations from research and policy, and operationalization for present study.

Approach	Description	Illustration	Operationalization for task design for present study
K-12 mathematics from an ``advanced standpoint''	Underlying structure and conceptual understanding of K-12 mathematics, at level of depth and rigor suitable for undergraduate mathematics	"What is required is a conceptual understanding of the material to be taught" (Baumert et al. 2010, p. 136)	Focuses on secondary mathematics topic. Topic is treated with rigor suitable for undergraduate course for mathematics concentrators. Reasoning on task uses definitions and concepts primarily from secondary mathematics.
Connections to disciplinary or higher grade-level content and practices	Deriving concepts of K-12 mathematics as applications of disciplinary or higher grade level results and practices	"For example, students … work to prove that there are no zero divisors for the real numbers. Many PSMTs may be comfortable saying that if $(x - 1)(x - 3) = 0$, then $x = 1$ and $x = 3$ are the solutions. However, without the above theorem they cannot explain why these are the only two solutions." (Murray and Star 2013, p. 1298)	Makes explicit reference to tertiary theorems, concepts, or methods of proof. Positions secondary mathematics results or phenomena as corollary of tertiary theorems, concepts, or methods of proof.
Mastery of conventional mathematics	Mathematics at the grade level and the equivalent of at least one grade level higher than the level of instruction.	"The [largest number of assessment items targeted the level of] intermediate (indicating content that is typically taught one or two grades beyond the highest grade the future teacher will teach)…" (Tatto et al. 2008, p. 37)	Exemplifies expectations described in standards documents, and addresses upper secondary mathematics.
Developing pedagogical content knowledge (PCK)	Connects content and pedagogy, is specific to teaching particular subject matter, and content knowledge is an important and necessary prerequisite	"Planning mathematical lessons", "Explaining or representing mathematical concepts or procedures" (Tatto et al. 2008, p. 39) "explain mathematical situations or to provide useful representations, analogies, illustrations, or examples to make mathematical content accessible to students" (Krauss et al., 2008, p. 876).	Exemplifies recurrent work of teaching secondary mathematics. The teaching practice is situated in a topic that is common to secondary mathematics. Involves selecting or constructing examples, explanations, analogies, metaphors, or demonstrations, where the audience is specified to be secondary school learners.

Mathematicians featured in the study of Leikin, Zazkis, and Meller (2017) wished for secondary teachers to learn additional mathematics content that could be used to supplement future lessons and possibly attract "strong" students to mathematics. In contrast, mathematicians Wu (2011) and Bass (2005) argued that teachers should foremost learn content that is used in recurrent work of teaching, rather than supplementary work. They stressed that teachers should learn definitions that could be plausibly used with students in the context of standard curricula.

Other mathematicians' reasons for wanting secondary teachers to have more mathematical and meta-mathematical knowledge coincide with advocacy for the various approaches (1) advanced standpoint, (2) tertiary connections, and (3) mastery. For instance, in line with (1), Peressini and Wu have authored rigorous treatments of secondary mathematics (Usiskin et al., 2003; Wu 2016). Cuoco and McCallum (2018) and mathematicians in Leikin, Zazkis, and Meller's (2017) study pointed to the importance of connecting university and secondary mathematics content and practices, consistent with (2). Informal surveys of mathematicians concerning secondary education suggest support for (1), (2), and (3) (Jackson and Rossi 1996; Madden 2000).

This literature suggests that there are a number of mathematicians who see the role of teacher education as providing mathematical and meta-mathematical substance and connections. This literature also suggests that there may be fewer mathematicians who would think to adopt approach (4), of emphasizing PCK in work with secondary teachers. Even Wu (2011), who argued stringently for the content of teacher preparation to be "relevant to teaching, i.e., [it] does not stray far from the material they teach in school" (p. 373), explicitly dismissed the potential efficacy of approach (4): "As this theory goes, teachers learn the mathematics better if it is taught hand in hand with pedagogy … teachers' lack of content knowledge is the more severe problem…. when I inject pedagogical issues into my teaching from time to time, the teachers are usually so preoccupied with learning the mathematics that the pedagogical discussion hardly ever takes place" (p. 381).

If the views discussed above were representative, then movement in capstone courses toward an approach that emphasizes PCK would require shifting how mathematicians see and value teachers' PCK, and tasks intended to elicit teachers' PCK.

An additional challenge to adopting an approach of developing teachers' PCK may come from the fact that, as Bass (2005) succinctly put it, "mathematics education is not mathematics" (p. 418). Working on tasks that require PCK may necessitate warrants that are not purely mathematical or meta-mathematical (Lai and Jacobson 2018). It may involve knowledge of student conceptions, research on which at least some mathematicians have excoriated (e.g., Wilf (2018)). Even at the undergraduate level, where instructors have the most experience, it may take time to develop sensitivity to students' difficulties and the ability to articulate one's pedagogical responses (Nardi et al., 2005). The tensions between warrants of scholarship in mathematics and mathematics education (Bass 2005; Dörfler 2003) may mean that tasks for developing teachers' PCK involve reasoning that mathematicians may neither typically practice nor think to teach.

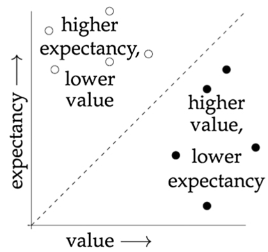

Fig. 1. Schematic representation of value and expectancy. Solid dots represent higher value and lower expectancy; for someone with such a profile, effective support for helping them to take actions consistent with values are likely to focus on increasing expectancy rather than increasing value.

3. Theoretical perspective

Wheatley (2005) observed that a key gap in teacher education research is "expectancy for learning to teach in new ways … from skillful use of new curricula or methods with *which they have little or no skill*" (p. 751, emphasis in the original). Decisions about how to use curricula or methods, whether with new or old ways, factor in how much a person *values* the goals that the curricula or methods might serve, as well as how well a person *expects* to do at carrying out the curricula or methods (Schoenfeld 2010). This stance is consistent with *expectancy-value theories*, broad motivational theories in educational psychology that connect actions, goals, values, and beliefs (e.g., Eccles and Wigfield 2002). A central aim of this paper is to show that the two constructs value and expectancy can be used to account for why developing teachers' PCK may not commonly be part of courses that mathematicians teach to teacher candidates.

Expectancy-value theory provides a competing explanation for the narrative that (i) one important reason people do not take actions that would help them is because they do not sufficiently value the outcomes of those actions; therefore, one must help the person understand why the outcomes are valuable. Expectancy-value theory finds that although value is an important factor, an equally important factor is expectancy: the confidence a person has that they would succeed. In this theory, (ii) people may not take actions that would help them because of a combination of how much they value the outcomes and how much they believe they would succeed at taking that action – and helping a person take beneficial actions includes helping the person gain confidence in taking those actions.

As an example, a person may not buy a gym membership for two reasons. The person may not value exercise, or the person may not expect to be good at exercise and fear the mockery of others in the gym. Expectancy-value theory holds that it is possible for a person to both value exercise *and* expect to be bad at exercising, and this may mean that the person does not buy a gym membership. In other words, even if values are strongly held, if expectancy is low, then a person may engage in tasks that oppose their values. Knowing the value of better habits does not itself support change. (In the schematic shown in Fig. 1, such a person would have a solid dot profile as opposed to an open dot profile.)

Similarly, secondary school teachers who choose not to teach proof may do so because they do not value proof and place a higher priority on procedural understanding or test-taking. Or, they may not teach proof because they do not believe they can effectively help students learn proofs.

Consider two alternative assumptions informing attempts to increase emphasis on developing PCK in courses taught by mathematicians. One assumption is that mathematicians do not value PCK. The alternative assumption is that mathematicians may need help because they lack confidence in their ability to teach for the development of PCK. If the mathematicians already value PCK, then convincing them of its value does no more good than exhorting to the unconfident gym member that exercise is important – in fact, such rhetoric may do harm by reinforcing self doubt. The goal of this study is to examine ways in which mathematicians may value or not value PCK, and reasons why mathematicians may have or not have confidence in teaching for the development of PCK. By understanding mathematicians' potential values and expectancies, we can more effectively determine strategies for increasing the emphasis on PCK.

Expectancy-value is not a common frame for research in undergraduate education (one exception is Ellis et al., 2016), but it has been used in mathematics education studies to explain students' choices and analyze teachers' strategies (Eccles 1983; Green 2002).

This study uses expectancy-value theory as a frame for understanding mathematicians' instructional decision making. Although studies have examined values and reflections of mathematicians (e.g., Hemmi 2010; Iannone and Nardi 2005), and their instructional patterns (e.g., Fukawa-Connelly 2012; Jaworski 2002; Weber 2004; Nardi et al., 2005), it is novel to simultaneously examine value, expectancy, and choice at the undergraduate level.

4. Study

4.1. Rationale

To examine value and expectancy for approaches to teacher education, mathematicians were asked during interviews to perform card sorts using a think-aloud protocol (Ericsson and Simon 1993). The cards in these card sorts represent approaches to teacher education as shown in Table 1, and were based on existing resources for teacher education when possible.[1] Card sorts with content-

Table 2
Participant characteristics.

Characteristic	Range
Experience teaching courses for prospective secondary teachers	0-10 years
prospective primary teachers	0-5 years
practicing secondary teachers	0-3 years
practicing primary teachers	0-3 years
prospective teachers of some level	1-10 years
"How good are you at teaching mathematics courses?" 1 (not at all) to 7 (very much)	5-7 (mode: 6)
"How good are you at teaching mathematics courses for teachers (prospective or practicing)?" 1 (not at all) to 7 (very much)	3-7 (mode: 4)
Institution type (number of participants): Doctoral (5), Masters (2), Baccalaureate (2)	
Other experience teaching (number of participants): Secondary teaching (1; 4 years), outreach (5)	

specific goals, content-generic goals, and tasks were used to elicit relative value and expectancy and also to examine consistency in each participants' concept of the approaches. Sorting on value simulated prioritizing. In teaching, one must select tasks and goals to accomplish in a limited amount of time (e.g., Lampert 2001). Even if many goals or tasks are valuable, it is often not possible to address all of them. Common wisdom provided to instructors across K-16 education suggests one should identify goals for the course as a whole, for each lesson, and then select or construct tasks that serve the goal (Sleep 2012). Thus one would expect consistency in how instructors perceived course-level goals, lesson-level goals, and tasks. At the same time, if inconsistencies arose, their reasons should be considered in explaining instructional choices.

4.2. Method

4.2.1. Participants

Mathematicians were recruited for an interview study as follows. Email invitations were sent to a list of 45 mathematicians who had participated in workshops on teacher education discussing mathematical knowledge for teaching; the author had co-facilitated some but not all of these workshops. The invitation specified that participants were sought for a "study on capstone courses". The invitation further specified that the investigator sought mathematicians who had or would, if given the opportunity, teach a capstone course for prospective secondary teachers. A total of 9 mathematicians agreed to participate. Characteristics of participants are summarized in Table 2, based on an initial internet survey.

4.2.2. Materials

The tasks and specific goals were selected based on the criteria described in the fourth column of Table 1. The generic goals were constructed to describe the purposes of each approach so as to be consistent with its conceptualization.

In a previous study on proofs for pedagogical purposes, some mathematicians evaluated a proof differently depending on the mathematical ideas emphasized (Lai and Weber 2014). Because the mathematical content of a task intended to elicit PCK can range from secondary content to beyond, these tasks are of two types: (a) those drawing on strictly secondary content and (b) those drawing on secondary mathematics from an advanced standpoint.

4.2.3. Procedure

The interviewer introduced the participants to the study by stating, "This study is about capstone courses, their purposes, and your teaching in them". Participants were then asked to do three activities, followed by a "wish list" question:

1. *Specific Goal Sort:* Four content-specific goals, shown in Fig. 2, were shown to participants representing the approaches.
2. *Task Sort.* Six cards containing tasks, shown in Fig. 3, were shown to participants representing the approaches. The tasks were named by colors (Pink, Orange, Yellow, Green, Blue, Violet). The Orange Task situates the Pink Task in a teaching situation. The Blue Task situates the definitions of function and graph in a teaching situation.
3. *Generic Goal Sort.* Four content-generic goals, shown in Fig. 2, were shown to participants representing the approaches.

At the end of the interview, participants were asked: "If you could make a wish list for resources for getting better at teaching math courses for teachers, what are some things that wish list would contain? What is inadequate about existing resources?"

During interviews, each task card was colored as named, while each goal card was white, to avoid participants identifying correspondences using the cards' physical appearance.

In card sorts, sorting can be done horizontally or vertically within a constrained area (e.g., a taped-off rectangle on an interview table). This study used both, for the purpose of asking participants to sort on two attributes: value and expectancy. To sort on value,

[1] Sources include Thames (2006) for the Green Task, and an instrument for mathematical knowledge for teaching developed by the Educational Testing Service © 2013, for the Yellow Task and Orange Task, which were adapted with permission. No endorsement of any kind by Educational Testing Service should be inferred.

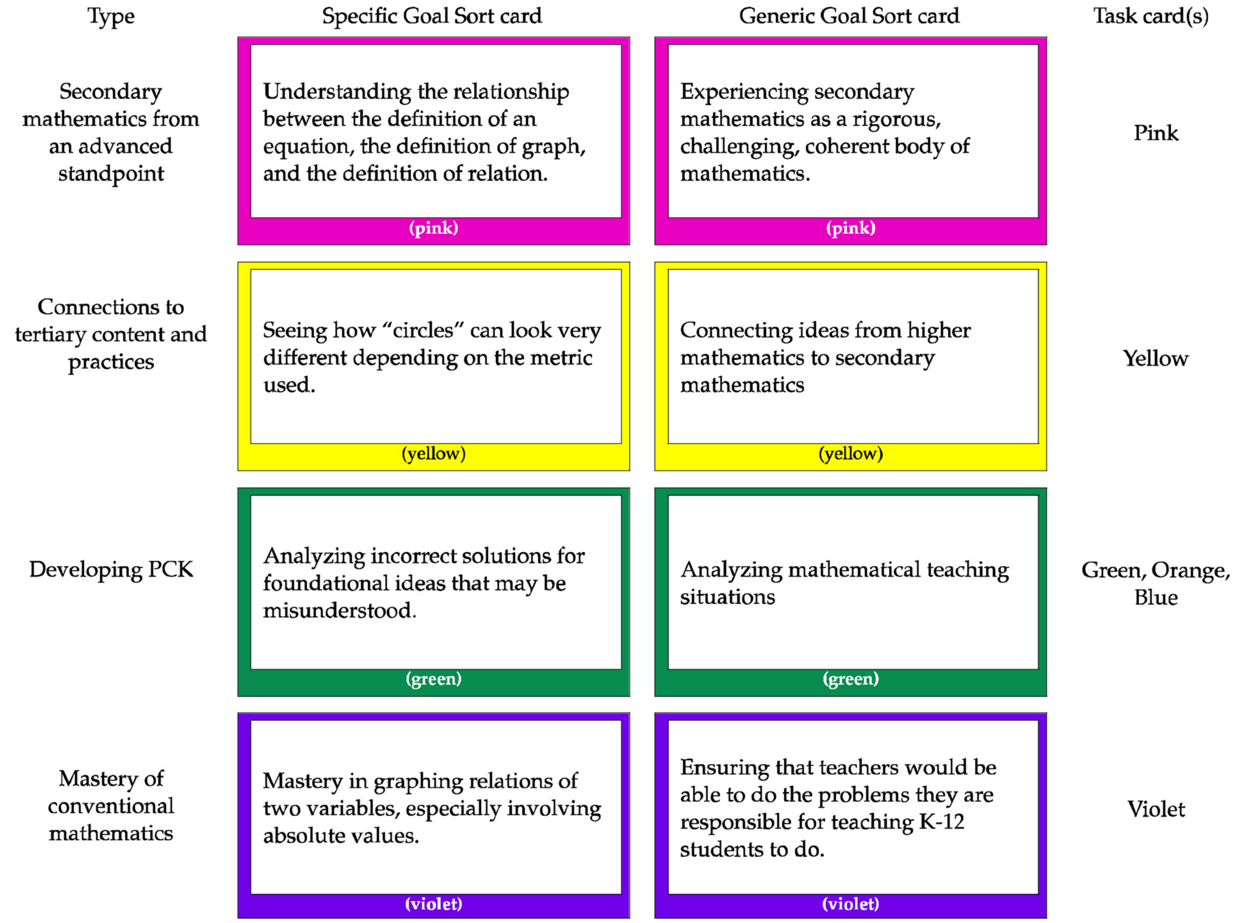

Fig. 2. Content-specific goals, content-generic goals, and their correspondence to tasks and approaches. The color noted in goal cards correspond to colors in Figs. 5 and 6.

participants were asked: "How well do these represent what secondary teachers should learn in their mathematical preparation? Place the card to the right if it is very valuable and to the left if it is not at all valuable." To sort on expectancy, participants were given the cards again and asked, "How confident are you that, if asked, you could create or learn to create opportunities in your teaching to help teachers to do well at these kinds of problems? Place the card toward the bottom if you are not at all confident. Place the card toward the top if you are very confident." Phrasings of questions on expectancy and value were adapted from Eccles et al. (1993). The value question was intended as aspirational, and participants' responses indicate that they understood the question as such, at times discussing the desire to achieve all goals. The protocol was piloted with advanced mathematics graduate students.

4.2.4. Analysis

To represent participants' card sorts, I used two-dimensional scatterplots, with value on the horizontal axis and expectancy on the vertical axis. A point indicates both value and the expectancy. The value is the horizontal coordinate and the expectancy is the vertical coordinate. For example, Fig. 4 shows participant Joe's card sort for six tasks. From this representation, we can read that Joe valued the Orange Task more than the Pink Task, but he was more confident that he could teach the Pink Task than the Orange Task.

For each card for each participant, the cards were assigned coordinates based on the location of the center of the card, as placed by the participant, where horizontal coordinates represented value and vertical represented expectancy.

Participants tended to place cards away from the left side (low expectancy) but not shy away from placing cards far on the right side (high expectancy). This may have been an artifact of warm-up questions to familiarize the participants with act of sorting. These warm-up questions included: (a) "How much do you like doing math while drinking coffee?", (b) "How much do you enjoy coming up with mathematical puns?", (c) "How confident are you that, if asked, you could create or learn to create opportunities for others to do math while drinking coffee?", (d) "How confident are you that, if asked, you could create or learn to create opportunities for others to make mathematical puns?" All 9 participants placed (d) as far to the left as possible, and most placed (c) as far to the right as possible, specifying that if "others" included fellow faculty members, this task would be very easy. Most placed (a) as high as possible, unless they did not like coffee, in which case they placed it as low as possible.

Interview transcripts from card sorts and wish lists were chunked into statements discussing orientations, goals, and resources

PINK TASK: Advanced Standpoint

Suppose $x \neq 0$. Prove that $x^0 = 1$. You may assume the additive law of exponents ($a^{b+c} = a^b a^c$ for all $a \in \mathbb{R}$, $b, c \geq 0$, and $b, c \in \mathbb{Z}$) and the definition that $a^1 = a$ for all $a \in \mathbb{R}$.

GREEN TASK: PCK-Secondary Content

Ms. Madison wants to pick one example from the previous day's homework on simplifying radicals to review at the beginning of today's class. Which of the following radicals is best for setting up a discussion about different solution paths for simplifying radical expressions?

(a) $\sqrt{54}$

(b) $\sqrt{72}$

(c) $\sqrt{120}$

(d) $\sqrt{124}$

(e) Each of them would work equally well.

Explain your reasoning.

ORANGE TASK: PCK-Advanced Standpoint

In Ms. Swain's Algebra I class, a student says the following.

> *I don't know why $x^0 = 1$. Is it just a convention? To me, it seems like it should be 0 because anything times 0 is 0.*

Write two different explanations that Ms. Swain can give to address what the student said. The explanations should address the underlying mathematics and be accessible to Algebra I students.

BLUE TASK: PCK-Advanced Standpoint

During a lesson on functions and their graphs, a student asks Mr. Loman:

> *The vertical line test talks about graphs and lines and points. But I thought we said that a function was something that has to do with rules about inputs and outputs. What do lines going through points have to do with rules about inputs and outputs?*

Write an explanation that Mr. Loman can give to address the student's question. The explanation should address the student's concerns about the connection between the underlying mathematics.

YELLOW TASK: Connections to Tertiary Content

During a lesson on exponentiation, Ms. Waller's students came across the expression $\left((-4)^{\frac{1}{2}}\right)^2$. Two students obtained different answers when they tried to evaluate this expression.

> **Anna**: *I got -4. I started with $(-4)^{\frac{1}{2}} = \sqrt{-4}$. And $\sqrt{-4} = 2i$. So $(2i)^2 = 4i^2 = -4$, and $((-4)^{\frac{1}{2}})^2 = -4$.*
>
> **Brenda**: *My answer was 4. I did $((-4)^{\frac{1}{2}})^2 = (-4)^{\frac{1}{2}\cdot 2} = (-4)^{2\cdot\frac{1}{2}} = ((-4)^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \sqrt{16} = 4$.*

Explain the apparent contradiction between Anna's and Brenda's answers in terms of a multi-valued exponential function.

PURPLE TASK: Mastery

Find three different pairs of functions g and h such that $g \circ h = (x+3)^2$.

Fig. 3. Tasks and their correspondence to approaches.

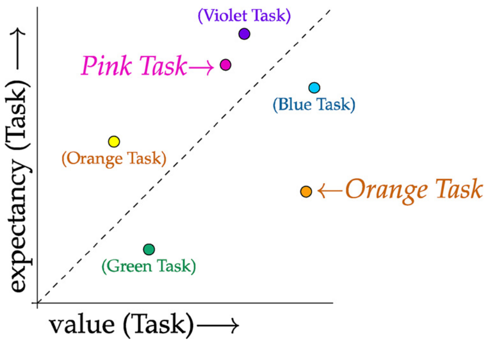

Fig. 4. Example placement of Task Sort cards, by participant Joe.

following Schoenfeld's (2010) model. The 9 participants made 184 such statements in total. Each statement was then coded for whether it discussed value or expectancy. Among these, 22 statements were coded as "neutral", meaning participants did not give any specific reason (e.g., "I think the orange card should be placed higher than the pink card" or "I feel confident about this"). Among the remaining, 85 statements addressed value ("I want to make sure this shows up because understanding where math comes from is important"), and 77 statements addressed expectancy ("I am confident because I've done this before"). The collection of statements with reasons were analyzed for themes using constant comparison (Strauss and Corbin 1994). Patterns noted in card placements were triangulated with interview statements.

5. Results

As discussed below, the participants generally valued tasks and goals for developing PCK more than other tasks and goals, and they were generally less confident about carrying out tasks and goals for developing PCK than other tasks and goals. There were few substantial differences between the placement of cards by type in the Task Sort, Specific Goal Sort, and Generic Goal Sort, although the differences between tasks and goals related to PCK and non-PCK were less pronounced in the Generic Goal sorts than other sorts. The overall trend indicated consistency in the individual participants' goals and tasks. The most detailed comments arose from the Task Sort, perhaps due to specificity, so they are often used to illustrate participant viewpoints.

I now describe how participants' interview responses addressed factors concerning value and expectancy, and how the analysis informed findings of this study. I focus on the most frequently occurring factors found, based on the number of statements made and the number of participants making them. Fig. 5 represents placements of cards for tasks for developing PCK compared to other task cards, aggregated across all participants. What is visible from Fig. 5 is that although there were some participants who were both confident about and valued tasks for developing PCK, in general, participants were less confident about tasks for developing PCK than other types of tasks *and* they valued tasks for developing PCK more than other types of tasks. Fig. 6 represents placements of cards for goals of developing PCK compared to other goal cards, aggregated across all participants. Tables 3 and 4 show complete lists of factors, and illustrate that the most frequently occurring factors were stated in favor of developing PCK rather than for other approaches.

5.1. Factors influencing value

The most frequently occurring factors for determining value were importance to teaching practice, proximity to secondary

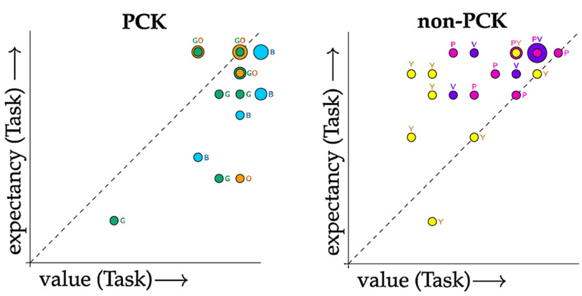

Fig. 5. Representation of task card placements, for all participants. Larger circles represent more participants placing a card in that location. Note that generally, participants expressed lower expectancy for tasks for developing PCK than other tasks, while also expressing higher value for tasks for developing PCK than other tasks. The color of each dot is denoted by the letters next to dots: P = Pink (Advanced Standpoint), O = Orange (PCK-Advanced Standpoint), Y = Yellow (Connections to Tertiary Content), G = Green (PCK-Secondary Content), B = Blue (PCK-Advanced Standpoint), V = Violet (Mastery).

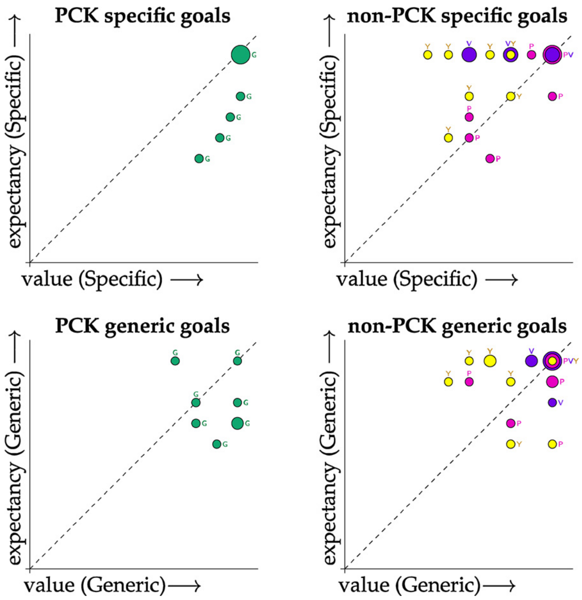

Fig. 6. Representation of generic and specific goal card placements, for all participants. Larger circles represent more participants placing a card in that location. The color of each dot is denoted by the letters next to dots: P = Pink (Advanced Standpoint), Y = Yellow (Connections to Tertiary Content), G = Green (PCK), V = Violet (Mastery).

Table 3
Ways mathematicians determined the value of a task or goal, and approaches valued.

Ways mathematicians determine value	Reason resulted in higher value for PCK	# statements (# distinct participants)
Importance to teaching practice	Yes	31 (9)
Proximity to secondary mathematics	Yes	21 (8)
Potential to enhance teachers' understanding of the nature or foundation of mathematics	No	13 (6)
How "mathematical" the task or goal is	No	5 (4)
Not knowing the content would be bad for teaching	No	6 (4)
Needs of particular teachers being taught	No	5 (3)
Priority for beginning teaching	Yes	4 (1)
Total statements		85 (9)

Table 4
Factors influencing mathematicians' expectancy.

Factors influencing mathematicians' expectancy	# statements (# distinct participants)
Difficulty of taking a PCK approach	22 (9)
Lack of experience / ability to draw on experience	34 (7)
Difficulty of explaining complex mathematics	15 (5)
Perceptions of prospective teachers' beliefs and attitudes	6 (3)
Total statements	77 (9)

mathematics, and enhancing mathematical understanding.

5.1.1. Importance to teaching practice

All 9 mathematicians made statements to this effect, with 31 statements in total about the value of a task or goal in terms of its importance to work that teachers do. These statements were often used to justify giving higher value to tasks and goals for developing PCK. Typical of this reasoning was Joe's thinking on the Specific Goal Sort. While moving the PCK card to represent higher value, he commented (italics mine):

> Because their students are going to misunderstand. There are going to be foundational ideas when they're teaching that they will need to convey and these ideas are often confusing. So *having them identify you know, what's wrong with this, this solution gives them experience*. *It gives them experience teaching.*

Joe is pointing out that teaching involves conveying ideas that students may have difficulty with, so the value of the goal for developing PCK is proximity to teaching practice. As Cantor reasoned in the Task Sort (italics mine):

> … if teachers can answer the pink one, they probably have a good answer to the orange one, but the orange one felt like more the situation a teacher would get…interact with, *that's more contextualized. So it got moved farther to the right [to represent higher value]."*

All participants valued the Orange Task more than the Pink Task, although not all provided explicit reasons why. One reason may be that, as Cantor pointed out, the Orange Task situates the mathematics of the Pink Task in teaching.

In each card sort, participants cited teaching as a reason to value tasks and goals, including those for an advanced standpoint. As Heidi commented in the Specific Goal Sort (italics mine), "I did put the 'analyze the incorrect solutions' slightly more left [representing less value] than the 'understanding relationship' and that's partially because I think *you need to understand the relationship first to really be able to analyze the incorrect pieces*." Heidi later commented that when a person can connect different concepts in secondary math, they can teach in a "flexible, creative way".

5.1.2. Proximity to secondary mathematics

Participants used closeness to secondary mathematics to justify greater value for tasks for developing mathematics from an advanced standpoint and PCK-advanced standpoint, and lesser value for connections to tertiary content. As Heidi commented in the Task Sort, "The detail needed for this particular question [connections to tertiary content] is probably beyond the necessary knowledge of somebody who is teaching algebra 1 or middle school math or geometry. … so that's kind of why I put them in the middle [meaning, lower than the task for mathematics from an advanced standpoint]." Participants generally valued tasks for developing mathematics from an advanced standpoint more than those developing connections to tertiary content. Two participants also used proximity to justify greater value for mastery.

5.1.3. Potential to enhance understanding the nature and foundation of mathematics

One main factor differentiated the value of PCK-secondary mathematics from that of advanced standpoint and PCK-advanced standpoint tasks and goals: whether the task captures "foundational" or "fundamental" mathematics. Typical of this reasoning was Dan:

> …the things that seem most important to me are understanding why things work the way they do … And some of these problems might illustrate these better than others, but I think the other questions get at deeper foundational issues.

Dan was pointing out that the less or more that foundational mathematics is addressed, the less or more important the task is.

Four participants noted that connections to tertiary content provide exposure to the nature of mathematics. However, these participants did not value these cards the most highly, hedging as to their importance. As Jim said in the Specific Goal Sort:

> We have to push them a little and expose them to a broader viewpoint. But what is the most important thing? Well they better know how to graph something and explain that to someone. Right?"

Two participants pointed out that while connections to tertiary content have a "neat factor", this was less important to them than mathematics of a more foundational nature.

5.2. Factors influencing expectancy

The most frequently occurring factors were demands of developing teachers' PCK and related past experiences.

5.2.1. Difficulty of developing teachers' PCK

All 9 participants noted the demands of teaching for developing teachers' PCK. First, participants commented on the difficulty of designing tasks for developing PCK. As Cantor observed of tasks for developing PCK (italics mine), "I tend to be good at recognizing, 'This is really good, these are interesting questions.' *But coming up with them, creating them myself, I think I tend to struggle*."

Eight of the participants commented in the wish list section that they wanted "more problems like the colored tasks".

It is unsurprising that the participants expressed difficulty in writing tasks for developing PCK. Although there is increasing knowledge on writing such tasks well and on how they function, the current state of the art requires multiple revisions involving different expertise (Gitomer et al. 2014). Writing tasks for developing PCK is an intensive process. Moreover, there are few available examples in commonly used resources for secondary teacher education that model either the writing process or the task design for instructors. Thus, these instructors predictably found it difficult to create tasks for developing PCK – even if they valued these tasks.

Assessing and understanding solutions to tasks for developing PCK also challenged participants. As Heidi commented about one such task, "It's hard seeing inside a student's head to be absolutely sure that you're communicating correctly. No matter how much assessment you do, we use words differently." Other participants commented that "proofs are something I can do", explaining that they were, in contrast, fluent with proofs. But tasks for developing PCK can ask for student-accessible explanations, or for describing how representations connect. Such explanations use language that may be less familiar to mathematicians, who are trained to communicate in formal language and proof.

Finally, one participant, Margaret, eloquently described the challenge of teaching prospective teachers to see from their future students' viewpoint: "You have to teach them how to look … They're viewing it from the right, the student's viewing it from the left; they're seeing mirror images, they're seeing different things, and they haven't even started to think about how this student is seeing it.” Although she felt sensitivity to students (Jaworski 2002) was necessary for developing PCK, she did not know how to cultivate prospective teachers’ sensitivity.

5.2.2. Related past experiences

Participants’ past positive experiences with tasks for developing PCK raised expectancy, and lack of experience lowered expectancy. For instance, Jim, on the generic goal of “analyzing mathematical teaching scenarios”, commented, “It would still be a challenge for me because of lack of experience. … I never took a course on classroom management or, or how to think about how to organize a course.” In contrast, Dan and Heidi, who had both had positive experiences teaching from Beckmann’s (2003) textbook for prospective primary teachers featuring many tasks for developing PCK, expressed confidence, even though the tasks developing PCK in the Task Sort were for a different course. As Dan reasoned, “I’m confident that I could create situations where they could do well or learn to do well, based on having done that in other courses.”

Previous experience also led to high expectancy on mastery tasks and goals, even though participants valued mastery less than developing PCK or mathematics from an advanced standpoint. They were confident because they had already taught courses with goals for developing mastery of conventional mathematical knowledge. These comments are consistent with findings in the expectancy-value literature; previous achievement-related experience, positive and negative, factor into a person’s expectancy of future success (Eccles and Wigfield 2002).

6. Discussion

The central argument of this paper is that if mathematics capstone courses for teachers, when taught by mathematicians, do not intentionally teach for developing teachers’ PCK, it is not necessarily because the mathematicians do not value PCK. Rather, mathematicians may lack confidence in teaching for the development of PCK, especially as compared to teaching for the development of more purely mathematical knowledge.

This paper opened by describing a programmatic issue in secondary teacher education, that theory and practice appear to indicate contrasting priorities. Namely, developing PCK is promising as an approach to mathematics teacher education, but developing PCK is not represented much in mathematics courses intended for teachers. One account for this would be that mathematicians do not value PCK, and that educators may seek to develop interventions based on this assumption. And, indeed, there may be mathematicians who fit this profile. However, expectancy-value theory provides another account: mathematicians may value PCK, but they are not as confident that they may be able to learn to teach for developing PCK well as other types of tasks. In this case, interventions based on promoting the value of PCK would be misguided. The qualitative analysis of this study provides a portrait of mathematicians fitting this latter profile. The participants nearly ubiquitously mentioned that they lacked confidence and resources with respect to the kinds of tasks they valued.

This study framed the examination of mathematicians’ instructional choices in a novel way, namely, using expectancy-value theory; in doing so, it addressed a gap noted by Wheatley (2005) about teacher education studies. The study addressed the question: *What factors influence expectancy and value for various approaches to capstone courses, particularly an approach that emphasizes the development of PCK*? The results suggest that relevant experience and resources impact expectancy. Personal struggles arose in participants’ discussion of taking an approach emphasizing the development of PCK. The only two participants who expressed greater confidence in this approach had prior experience doing so at primary level.

This study has two main implications for shifting the mathematics instruction that prospective secondary teachers experience, particularly for mathematicians who may have neither teacher education background nor prior experience teaching courses that emphasize the development of PCK. First, the literature on expectancy-value theory suggests that a person is most likely to pursue a goal when the goal is valued, it generates high expectancy, and there are resources that a person can use to break down how to implement the goal (Gollwitzer and Sheeran 2006). When expectancy and resources are not in place, the person may make choices counter to the goal – even if the person highly values the goal. Here, the implication is: mathematicians may actually value developing teachers’ PCK, but they do not enact tasks eliciting PCK, because they lack the confidence and resources to do so.

Second, these mathematicians valued tasks that captured the work of teaching and where the content was close to secondary mathematics. These are the same reasons that mathematics educators have cited for tasks to use in mathematics teacher education: tasks should provide teachers practice with coordinating mathematical and pedagogical reasoning, and the content should give direct insight into that which the teachers teach (Hill, Sleep, Lewis, and Ball 2007; Hoover et al., 2016; Howell et al., 2015; McCrory et al. 2012). The implication here is that some mathematicians and mathematics teacher educators may not only share the value of developing PCK, they may also have come to those values similarly.

6.1. Limitations

I note two limitations of the study. First is the sampling and identity of the interviewer. The participants were recruited from attendees of workshops on mathematical knowledge for teaching; the interviewer is a mathematics educator and was a co-facilitator of some of the workshops. It may have been that the experience of participants in the workshop made them realize the importance of PCK as well as the challenge for teaching it. And it may have been that had mathematicians been interviewed by a member of their department, such as their department chair, that they may have expressed different opinions as to value and expectancy. It is worth noting that at the time of the interview, the workshops had taken place more than 2 years ago. Although there may be effects from workshop experiences, the comments of the participants largely centered on their own experiences teaching, suggesting that any difficulties expressed were reflected in their actual teaching. It may be that had a department chair conducted the interviews, that the participants would have expressed less value for an approach developing PCK and more value for the other more strictly mathematical approaches. However, given the relative autonomy of mathematicians in course instruction, it is likely that if the mathematicians do value an approach developing PCK, and they felt able to enact it, they would still pursue an approach developing PCK.

Related to this issue is the sample size of this study. Nonetheless, this study is valuable in documenting a particular phenomenon: developing teachers' PCK can be highly valued *and* generate mixed expectancy, and expectancy can be informed by previous experience. There is reason to believe that this result can be applied productively to mathematicians who teach capstone courses for secondary teachers. Policy documents such as *The Mathematical Education of Teachers, II* (CBMS 2012) that are written by consortia involving mathematicians do describe the value of PCK for K-12 education. The mathematicians who participated in this study represent ones who have the trait of wanting to develop teachers' PCK. Helping this population develop more confidence may promote the development of mathematician leaders who can serve as resources for other mathematicians who seek to incorporate an approach that fosters teachers' PCK.

The second limitation is scope. The participants examined only 14 tasks and goals. It is possible that a different set of tasks or goals would have elicited different value and expectancy. This study identified some values that were used in prioritizing among approaches; future studies could vary the scope and also examine alignment of values, expectancy, and instructional decisions in their actual teaching.

6.2. "Something I can do."

One motivation for studies of expectancy and value in teaching is to identify why teachers feel that they can or cannot carry out a valued but unfamiliar teaching approach. Efforts to shift instruction towards that approach are then designed around underlying causes of low expectancy. Mathematicians in this study were concerned with their ability to give constructive feedback, cultivate teachers' sensitivity to students, and create tasks for developing teachers' PCK.

Looking forward, I lay out implications for research and practice to support approaches emphasizing PCK at the secondary level. To frame this discussion, I return to participants' observation that, in contrast to tasks eliciting PCK, "proofs are something I can do." What would it take for mathematicians to perceive tasks eliciting PCK as "something I can do"? And, further, "something I can teach"? Although the participants' observations were about doing proofs rather than teaching proofs, it stands to reason that one must know something before one can teach it – and that these participants were saying that did not feel they had the requisite knowledge to develop teachers' PCK.

As the participants in this study alluded, teaching to develop teachers' PCK can place a considerable demand on mathematicians who must organize their course to be conducive to an unfamiliar kind of task, scaffold prospective teachers' thinking on these tasks, respond to ideas that may be off-track, and distill mathematical and pedagogical warrants in explicit and coordinated ways. For more mathematicians to teach in ways that emphasize PCK, resources are needed to suggest tasks for developing teachers' PCK, help mathematicians themselves develop PCK for teaching at the secondary level, and develop mathematicians' knowledge for developing secondary teachers' PCK.

The demand for such multi-tiered knowledge development requires coordinating research and practice efforts. Studies of teacher development at the school level suggest that interventions are more likely to be successful when professional developers and curriculum designers can work from knowledge of how learners' conceptions develop, noticing what learners say and do as evidence of development, and how activity features elicit and shape learners' reasoning (Davis and Krajcik 2005). Applying these results to the context of capstone courses, mathematics teachers educators need knowledge of how teachers' PCK develops, how to notice what teachers say and do, and how activity features elicit or shape teachers' mathematical and pedagogical reasoning.

Thus one implication is the need for more fine-grained knowledge about reasoning on tasks developing PCK. Although there is some existing research on how task design can influence reasoning (e.g., Lai and Jacobson 2018), this area is still in foundational stages. There is not much known beyond case studies.

To create an analogy, mathematicians are trained to use *mathematical* and *meta-mathematical* knowledge, and mathematicians use meta-mathematical knowledge to tackle problems in unfamiliar mathematical domains, as well as to analyze and improve students' capacity to do mathematical proof. The greater confidence of the two participants with prior experience with an approach emphasizing PCK at the primary level suggests that they saw potential similarities between primary level and secondary level, despite topical differences. Articulating "meta-PCK" understandings may go a long way in helping instructors develop the confidence to take on an approach emphasizing PCK, as well as creating language for noticing how PCK is used.

A second implication is that educators should use existing theories for how knowledge for teaching develops (e.g., Silverman and Thompson 2008; Nardi et al., 2005) to design resources for capstone courses for secondary teachers – and simultaneously test the accuracy of these theories for both secondary teachers *and* mathematicians. The research on development of knowledge for teaching is in relatively nascent stages, with results primarily based on samples that are restricted in scope and size, or proposed from mostly theoretical standpoints. Research on the instruction of capstone courses provides an opportunity to build on existing research while generating new knowledge to improve the instruction of teachers.

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